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Negative correlation properties in graphs or: How I learned to stop looking for the edge e among spanning trees already containing f and instead look among all spanning trees.

Alejandro Erickson

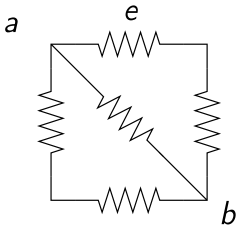
Master's thesis work while at the
Department of Combinatorics and Optimization
University of Waterloo

April 11, 2010

Outline

Kirchhoff's law

Electrical network of resistors, each with conductance y_g.



e has resistance r_e,
conductance $y_e = \frac{1}{r_e}$

Kirchhoff's law gives a formula for the conductance between nodes a and b.

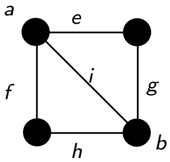
Model the network as a graph G (shocking!) and let T be the generating polynomial for spanning trees of G.

$T(G) =$  $+$  $+$  $+$ 

$+$  $+$  $+$  $+$ 

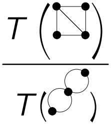
$$= y_e y_f y_h + y_e y_f y_g + y_e y_g y_h + y_f y_g y_h$$

$$+ y_e y_h y_i + y_f y_g y_i + y_e y_f y_i + y_g y_h y_i$$

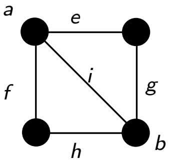


Let G/ab be G with nodes a and b identified.

Kirchhoff's Law



$$\mathcal{Y}_{ab} = \frac{T(\cdot)}{T(\cdot)} = \frac{T(G)}{T(G/ab)}$$



Rayleigh monotonicity

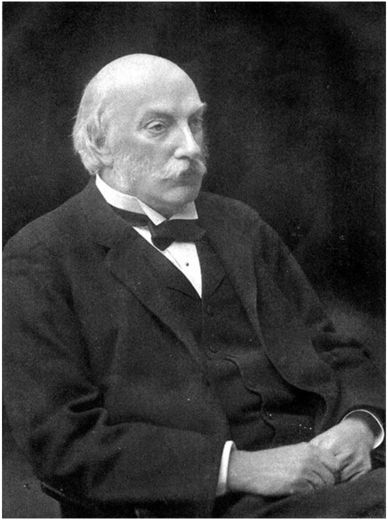
Lord Rayleigh (1842-1919) observed that increasing the conductance of any edge should not decrease the effective conductance of the whole network.

$$\mathcal{Y}_{ab} = \frac{T(G)}{T(G/ab)}$$

$\mathcal{Y}_{ab}$ is non-decreasing in the direction of every variable y_e.

For any edge e, we have

$$\frac{\partial}{\partial y_e}\mathcal{Y}_{ab} \geq 0.$$



Three bits of notation:

- T^g denotes evaluation at $y_g = 0$. $T^g = T(G \setminus g)$.
- T_g denotes partial derivative w.r.t y_g. $T_g = T(G/g)$.
- Add an edge, $G \leftarrow G + f$ where $f = ab$.

[1] old G

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Now, **Kirchhoff's law** is[1]

$$\mathcal{Y}_{ab} = \frac{T(G)}{T(G/ab)} = \frac{T^f}{T_f}$$

and the **Rayleigh property** is that

$$\frac{T_e^f T_f - T^f T_{ef}}{(T_f)^2} \geq 0,$$

for each distinct pair of edges e and f and positive y_gs

[1] old G

Forget all that stuff about electrical networks.

Selecting trees

Notice that

- $y_g T_g$ are the spanning tress containing g
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If we restrict ourselves to trees *already* containing f, then the chances our tree contains e are

$$\frac{y_e y_f T_{ef}}{y_f T_f}.$$

Equivalent conditions

Obvious but important:

T consists of those terms not containing g and those containing g. That is

$$T = T^g + y_g T_g$$

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Back to the Rayleigh condition

$$\begin{aligned}
& (T_e)T_f - (T)T_{ef} \\
&= (T_e^f + y_f T_{ef})T_f - (T^f + y_f T_f)T_{ef} \\
\text{(Rayleigh)} \quad &= (T_e^f)T_f - (T^f)T_{ef} \geq 0
\end{aligned}$$

So what!?!

We showed that

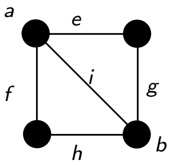
$$T_e T_f - T T_{ef} = T_e^f T_f - T^f T_{ef} \geq 0.$$

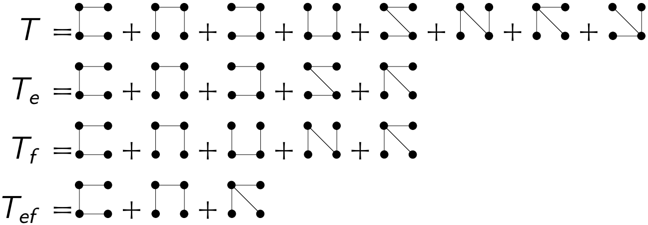
The missing piece

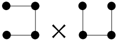
$$\begin{aligned} T_e T_f - T T_{ef} &\geq 0 \text{ if and only if} \\ y_e y_f (T_e T_f - T T_{ef}) &\geq 0 \text{ if and only if} \\ \frac{y_e T_e}{T} &\geq \frac{y_e y_f T_{ef}}{y_f T_f}. \end{aligned}$$

So the chances of selecting a spanning tree with e are not increased by choosing among those *already* containing f!

Time for an example





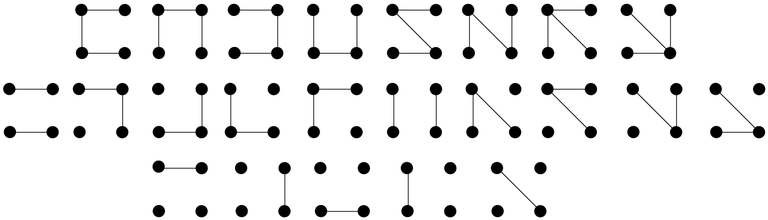
and $y_e y_f (T_e T_f - T T_{ef}) = y_e y_g y_h \times y_f y_g y_h =$ 

Where is the proof?

- The classical proof using electrical networks is printed in Grimmett's book.
- The most often cited proof is due to Brooks Smith Stone and Tutte (1940).
- A stronger property was shown by Choe and Wagner (2006).
- A combinatorical (bijective) proof is given by Cibulka, Hladky, LaCroix and Wagner (2008).

So if this stuff has been done over at least four times, what's all the fuss?

Mathematicians love variations!



Let's replace T by the **spanning forests**, F.

This was proposed in print in the early 90s.
Considerable evidence has been published but, as of yet, no proof that

$$F_e F_f - F F_{ef} \geq 0,$$

for positive y_gs and pair of distint edges e and f.

A “weaker” version

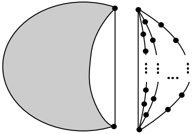
Special case of Rayeligh, $F_e F_f - F F_{ef}$

- Set each y_g to 1.
- ie, choose spanning forest uniformly at random.

Special case $\equiv$ Rayleigh

(independently: Cocks and E., 2008)

- All graphs are forest Rayleigh iff
- all graphs satisfy the special case.



proof idea: Suppose a graph is not Rayleigh, then $F_e F_f - F F_{ef} < 0$ for certain y_gs. Replace edges by certain disjoint paths to create a graph that is not negatively correlated.

Evidence for the conjecture

– Small graphs
are negatively correlated
(Grimmett, Winkler, 2004).

– Two-sums
of Rayleigh graphs
are Rayleigh (Wagner,
Semple, Welsh 2008)

$\longrightarrow$ Smaller graphs are
Rayleigh (E., Wagner, 2008)

$\longrightarrow$

Series parallel graphs are
Rayleigh (E., Wagner, 2008)

SOS conjecture (Wagner)

The spanning forest Rayleigh difference,

$$\Delta F\{e,f\} = F_e F_f - F F_{ef}$$

is a sum of monomials times squares of polynomials,

$$\Delta F\{e,f\} = \sum_S \mathbf{y}^S A(S)^2.$$

SOS conjecture (Wagner)

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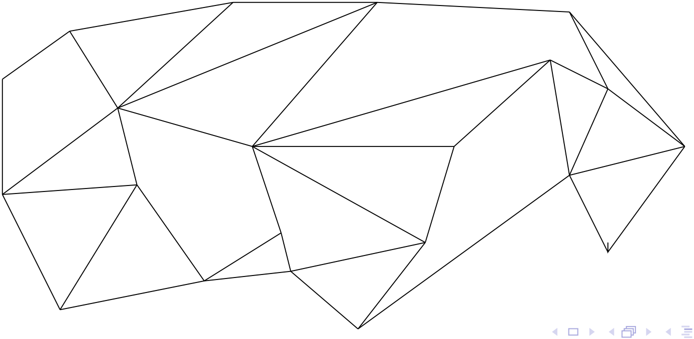
The Rayleigh property, $F_e F_f - F F_{ef} \geq 0$ for positive y_gs, follows immediately.
One major hangup: the signs of the terms in $A(S)$ are unknown.

S-sets and A-sets

$$\Delta F\{e,f\} = \sum_S \mathbf{y}^S A(S)^2.$$

- An S-set is a set of edges S so that $S \cup \{e,f\}$ is contained in a cycle.
- The A-sets of S are those spanning forests A so that $A \cup \{e,f\}$ contains a unique cycle which contains S.



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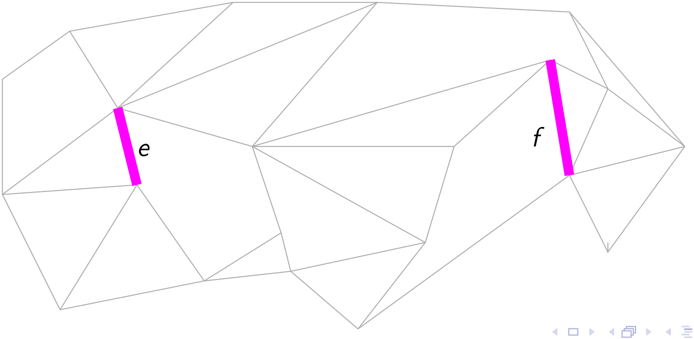

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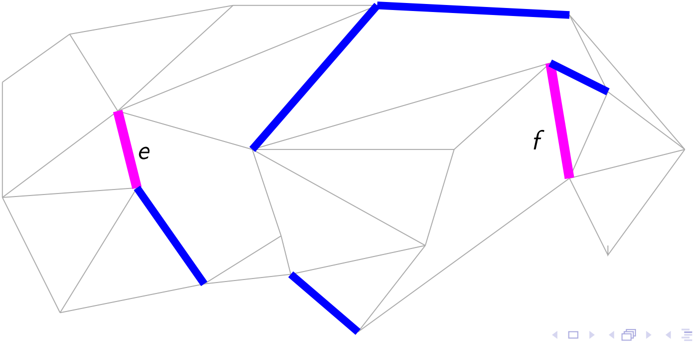

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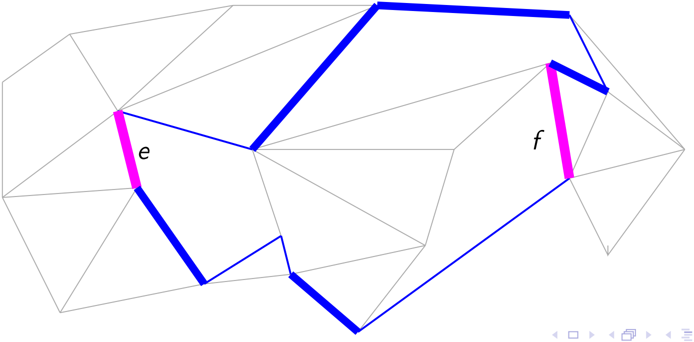

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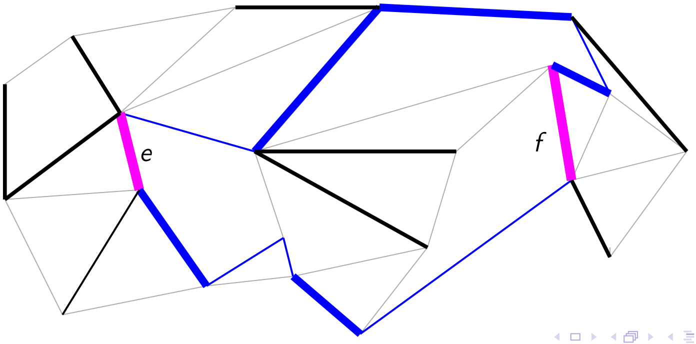


Given an S-set, S with $S \cup \{e, f\}$ contained in a cycle C,

$$A(S) = \sum_{A} c(S, e, f, C)\mathbf{y}^{A-S}.$$

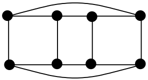
There they are! The signs $c(S, e, f, C)$. And there are MANY of them.

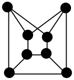
Testing on small graphs

Wagner had some guesses for the signs and we tested

$$\sum_S \mathbf{y}^S A(S)^2 = F_e F_f - F F_{ef}$$

in Maple, for graphs up to 7 vertices.
He also found signs that worked for the cube and Möbius ladder on 8 vertices.



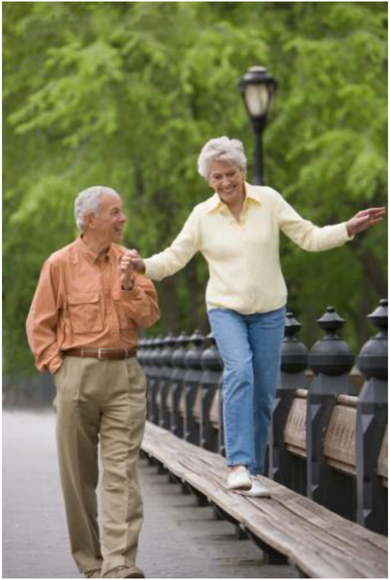


Necessary conditions

Next, we "show" the SOS-conjecture should hold for two sums and that it does hold for series parallel graphs.

Series parallel graphs and 2-sums

My presentation



The details

Suppose $G = H \oplus_g K$ and let C be a cycle of G. Then either C is contained in $H - g$ or $K - g$ or $C = C_H \cup C_K - g$ for cycles through g in H and K.

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Facts about 2 sums and $\Delta F\{e, f\} = F_e F_f - F F_{ef}$

- If $e \in H$ and $f \in K$, then $\Delta F(G)\{e, f\} := \Delta F(H)\{e, g\} \Delta F(K)\{f, g\}$
- If $e, f \in H$, then $\Delta F(G)\{e, f\} = F(K)^2 \Delta F(H)\{e, f\}$

The Rayleigh difference factors over the factors of the 2-sum.

How does the SOS-form factor?

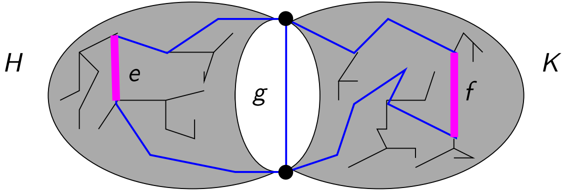
$$\sum_S \mathbf{y}^S A(S)^2$$

is all about the cycles of G, through e and f

If $e \in H$ and $f \in K$, these cycles come from $C = C_H \cup C_K - g$.

$$\Delta F(G)\{e,f\} := \Delta F(H)\{e,g\} \Delta F(K)\{f,g\}$$

In the same way, A-sets of G come from A-sets of H and K, so the SOS-form factors.

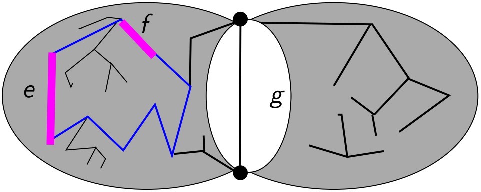


How does the SOS-form factor?

Show $\sum_{S} \mathbf{y}^{S} A(S)^2 = F(K)^2 \Delta F(H)\{e,f\} = F(K)^2 \sum_{S_H} \mathbf{y}^{S_H} A_H(S_H)^2$

If $e, f \in H$, we sum over

- S-sets in H not containing g.
  - careful that A-sets of H and forests of K do not form extra cycles in G.



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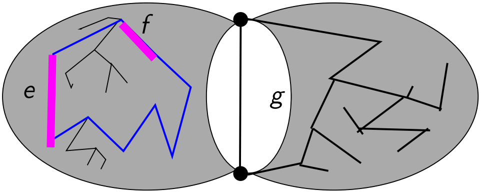

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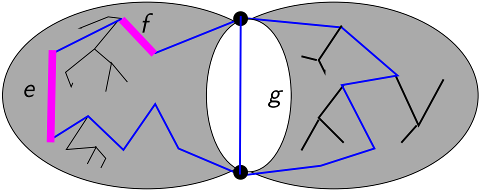
If $e, f \in H$, we sum over

- S-sets in H not containing g.
  - careful that A-sets of H and forests of K do not form extra cycles in G.
- S-sets containing g in H and edges of K.
  - snag! The forests we use from K need to make a unique cycle with g and satisfy another SOS form.



$(K^g - K_g)K_g = \sum_Q \mathbf{y}^Q B(Q)^2$?

Series parallel graphs.

Δ-SOS

$$\Delta F\{e,f\} = F_e F_f - F F_{ef} = \sum_S \mathbf{y}^S A(S)^2$$

Φ-SOS

$$\Phi F\{g\} = (F^g - F_g)F_g = \sum_Q \mathbf{y}^Q B(Q)^2.$$

If K is series parallel then it is Φ-SOS.

Hope for 2-sum

K is Δ-SOS by inductive hypothesis. Can we show that if K is Δ-SOS, then it is Φ-SOS?
This reduces to yet a third "SOS" form (see paper).

Summary

- Goal: Prove

$$\Delta F\{e,f\} = F_e F_f - F F_{ef} \geq 0$$

- Method: Prove

$$F_e F_f - F F_{ef} = \sum_S \mathbf{y}^S A(S)^2$$

- Next step: Prove

$$\Phi F\{g\} = (F^g - F_g) F_g = \sum_Q \mathbf{y}^Q B(Q)^2.$$

Many thanks to David Wagner and my classmates from Waterloo for their ideas and encouragements.